Differential Geometry IV: General Relativity

13.1 In this exercise, we will examine the relationship between the intrinsic and extrinsic geometry of a smooth spacelike hypersurface Σ in a Lorentzian spacetime $(\mathcal{M}, g)$ (the same relations in fact hold more generally for non-degenerate metrics of arbitrary signature).

Let us denote with $\bar{g}$ the induced Riemannian metric on Σ and with $\hat{n}$ the future directed timelike unit normal of Σ. For any $p \in \Sigma$, we will define the orthogonal projection operators $\pi^{\perp} : T_p\mathcal{M} \to \langle \hat{n} \rangle$ and $\pi^{\top} : T_p\mathcal{M} \to T_p\Sigma$ by

$$\pi^{\perp}(X) \doteq -g(X, \hat{n})\hat{n} \quad \text{and} \quad \pi^{\top}(X) \doteq X - \pi^{\perp}(X)$$

(you can readily check that this is indeed an orthogonal decomposition; note that $\pi^{\top}$ projects onto the tangent space of the hypersurface Σ). We will also set $\Gamma(\mathcal{M}, \Sigma)$ to be the set of vector fields which are tangential to Σ, i.e.

$$\Gamma(\mathcal{M}, \Sigma) \doteq \Big\{ X \in \Gamma(M) : \, X|_p \in T_p\Sigma \, \text{for all } p \in \Sigma \Big\}$$

(note that $\pi^{\perp}(X) = 0$ at any point $p \in \Sigma$ for every $X \in \Gamma(\mathcal{M}, \Sigma)$).

(a) Show that if $X, Y \in \Gamma(\mathcal{M}, \Sigma)$, then $[X, Y] \in \Gamma(\mathcal{M}, \Sigma)$. Show also that any vector field $Z \in \Gamma(\Sigma)$ on Σ can be extended (non-uniquely) to a vector field $Z \in \Gamma(\mathcal{M}, \Sigma)$ *(Hint: Use local coordinates in which Σ coincides with the level set $\{x^0 = 0\}$).*

(b) For any $X, Y \in \Gamma(\mathcal{M}, \Sigma)$, we will define along Σ:

$$\bar{\nabla}_X Y \doteq \pi^{\top}\big(\nabla_X Y\big).$$

Show that $\bar{\nabla}$ is the Levi-Civita connection of the induced metric $\bar{g}$ on Σ (assuming that vector fields in $\Gamma(\Sigma)$ are extended to vector fields $\Gamma(\mathcal{M}, \Sigma)$ as in the previous question). *(Hint: Check that $\bar{\nabla}$ satisfies the defining properties of the Levi-Civita connection.)*

(c) **The second fundamental form.** For any $X, Y \in \Gamma(\mathcal{M}, \Sigma)$, we will define along Σ:

$$k(X, Y) \doteq g(\nabla_X \hat{n}, Y).$$

Show that k is a symmetric $(0, 2)$-tensor on Σ. Show also that

$$k(X, Y)\hat{n} = \pi^{\perp}(\nabla_X Y) \quad \text{for all } X, Y \in \Gamma(\mathcal{M}, \Sigma),$$

so that

$$\nabla_X Y = \bar{\nabla}_X Y + k(X, Y)\hat{n}. \tag{1}$$

(c) **The Gauss equation.** Prove that, along Σ, the following identity holds for any $X, Y, Z, W \in \Gamma(\mathcal{M}, \Sigma)$:

$$g\big(R(X, Y)Z, W\big) = \bar{g}\big(\bar{R}(X, Y)Z, W\big) + k(Y, Z)k(X, W) - k(X, Z)k(Y, W),$$

where $\bar{R}$ is the Riemann curvature tensor of $\bar{g}$. *(Hint: In the definition of the Riemann curvature tensor R for g, use the decomposition* (1)*.)*

(d) **The Codazzi equation.** Show that

$$g\big(R(X,Y)Z,\hat{n}\big) = (\nabla_Y k)(X,Z) - (\nabla_X k)(Y,Z).$$

13.2 The constraint equations. Let $(\mathcal{M}^{n+1}, g)$ be a spacetime satisfying the Einstein equation

$$Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

(where T is the energy momentum tensor of some matter field; you can assume that it is simply a symmetric $(0,2)$-tensor on $\mathcal{M}$) and let Σ^n be a spacelike hypersurface of $\mathcal{M}$ with future directed timelike unit normal $\hat{n}$. As in the previous exercise, let $\bar{g}$ be the induced metric on Σ and k its second fundamental form (with respect to $\hat{n}$).

(a) **The Hamiltonian constraint equation.** Show that, along Σ:

$$\bar{R} - \|k\|_{\bar{g}}^2 + (\mathrm{tr}_{\bar{g}} k)^2 = 16\pi T(\hat{n},\hat{n}),$$

where $\bar{R}$ is the scalar curvature of the induced metric $\bar{g}$ and $\mathrm{tr}_{\bar{g}} k \doteq \bar{g}^{ij}k_{ij}$. *(Hint: At any point $p \in \Sigma$, pick an orthonormal frame $\{e_\alpha\}_{\alpha=0}^n$ for $T_p\mathcal{M}$ with $e_0 = \hat{n}$ and use the Gauss equation from Exercise 13.1; note that, in such a frame, $Ric(e_\mu, e_\nu) = -g\big(R(e_\mu,e_0)e_0,e_\nu\big) + \sum_{i=1}^n g\big(R(e_\mu,e_i)e_i,e_\nu\big)$ and $\bar{R} = \sum_{i,j=1}^n \bar{g}\big(\bar{R}(e_i,e_j)e_j,e_i\big)$.)*

(b) **The momentum constraint equation.** Show that, for any $X \in \Gamma(\mathcal{M},\Sigma)$,

$$\Big(\mathrm{div}_{\bar{g}}\big(k - (\mathrm{tr}_{\bar{g}}k)\bar{g}\big)\Big)(X) = 8\pi T(\hat{n},X)$$

where, in local coordinates on Σ, $(\mathrm{div}_{\bar{g}}B)_i \doteq \bar{g}^{ab}\nabla_a B_{bi}$ for any $(0,2)$-tensor B. *(Hint: Use the Codazzi equation.*

Remark. The quantities $\rho = T(\hat{n},\hat{n})$ and $J = T(\cdot,\hat{n})$ appearing above are the so-called *mass density* and *momentum density* of the matter field with energy momentum tensor T. A realistic matter field satisfies the *dominant energy condition*

$$\rho \geqslant \|J\|_{\bar{g}}.$$

You can check that this condition is indeed satisfied in the case of the energy-momentum tensor of a scalar wave ψ (i.e. when $T = d\psi\otimes d\psi - \frac{1}{2}g^{-1}(d\psi,d\psi)g$).

13.3 In this exercise, we will use the conformal compactification of Minkowski spacetime to deduce some decay estimates for solutions to the linear wave equation, as well as global existence results for solutions to certain nonlinear wave equations. This method was first introduced by Christodoulou in '86.

(a) Recall, as we saw in class, that, expressed with respect to the usual double-null coordinate system (u,v,θ,φ) (with $u=t-r, v=t+r$) on Minkowski spacetime $(\mathbb{R}^{3+1},\eta)$, the map $\mathcal{F}:(u,v,\theta,\varphi)\to(U(u),V(v),\theta,\varphi)$ with

$$U(u)=\text{Arctan(u)},\quad V(v)=\text{Arctan(u)}$$

maps $(\mathbb{R}^{3+1},\eta)$ *conformally* into a pre-compact subset of the Einstein cylinder $(\mathbb{R}\times\mathbb{S}^3,g_E)$. Recall that, with respect to the usual polar coordinates (T,R,θ,φ) on the cylinder $\mathbb{R}\times\mathbb{S}^3$, the metric g_E is the usual product metric

$$g_E=-dT^2+g_{\mathbb{S}^3}=-dT^2+dR^2+\sin^2 R(d\theta^2+\sin^2\theta d\varphi^2)$$

and the coordinates (U,V) are related to (T,R) by

$$T=V+U,\quad R=V-U.$$

Show that

$$\mathcal{F}^*g_E=\Omega^2\eta,\quad \Omega(u,v)=2\cos(U)\cos(V)$$

(we have seen this in class; it is worth redoing the calculations for what follows). Describe the domain $\mathcal{F}(\mathbb{R}^{3+1})$ with respect to the (U,V,θ,φ) and (T,R,θ,φ) coordinates and identify the future and past null infinities $\mathcal{I}^{\pm}$, the future and past timelike infinity $\iota^{\pm}$ and the spacelike infinity ι^0. Show that $\mathcal{F}\big(\{t=0\}\big)$ is dense in the hypersurface $\{T=0\}$ of $\mathbb{R}\times\mathbb{S}^3$. Verify that the map $\mathcal{F}$ is smooth at $r=0$ (where the double null coordinate system breaks down; you might want to switch to Cartesian coordinates for this task).

Remark. In what follows, we will identify $\mathbb{R}^{3+1}$ with its image $\mathcal{F}(\mathbb{R}^{3+1})$ in the Einstein cylinder via the map $\mathcal{F}$, so that (u,v,θ,φ) and (U,V,θ,φ) can be thought of as coordinates systems on either $\mathbb{R}^{3+1}$ or $\mathcal{F}(\mathbb{R}^{3+1})$. In this spirit, if h is a function, say, on $\mathbb{R}^{3+1}$, we will also denote by h the function $h\circ\mathcal{F}^{-1}$ on $\mathcal{F}(\mathbb{R}^{3+1})$.

(b) Let $\phi:\mathbb{R}^{3+1}\to\mathbb{R}$ be a smooth function satisfying

$$\square_\eta\phi=F.$$

Show that the function

$$\tilde{\phi}=\Omega^{-1}\phi$$

on $\mathcal{F}(\mathbb{R}^{3+1})\subset\mathbb{R}\times\mathbb{S}^3$ solves the conformal wave equation

$$\square_{g_E}\tilde{\phi}-\tilde{\phi}=\Omega^{-3}F.$$

Remark. Note that, in general, the operator $P_g(f)\doteq\square_g f-\frac{n-1}{4n}R_g f$ on $(\mathcal{M}^{n+1},g)$ is conformally invariant, in the sense that $e^{\frac{n+3}{2}h}P_{e^{2h}g}f=P_g(e^{\frac{n-1}{2}h}f)$.

(c) Assumt that $\phi : \mathbb{R}^{3+1} \to \mathbb{R}$ solves

$$\Box_\eta \phi = 0$$

with smooth and compactly supported initial data at $t = 0$. Show that $\tilde{\phi}$ (defined as before) solves

$$\Box_{g_E} \tilde{\phi} - \tilde{\phi} = 0$$

with compactly supported initial data at $T = 0$. Show that $\tilde{\phi}$ is uniformly bounded on $\mathcal{F}(\mathbb{R}^{3+1})$ *(Hint: Use the general well-posedness theorem for linear wave equations on* $\mathbb{R} \times \mathbb{S}^3$*).* Deduce then that ϕ satisfies the following decay bounds on $\mathbb{R}^{3+1}$:

$$|\phi| \lesssim \frac{1}{(1+|u|)(1+|v|)}.$$

In particular, with respect to the Cartesian coordinates, $|\phi| \lesssim \frac{1}{t}$ as $t \to +\infty$ along the null cones $t - |x| = const$ and $|\phi(t,x)| \lesssim t^{-2}$ as $t \to +\infty$ for fixed x.

(d) For any function $f : \mathbb{S}^3 \to \mathbb{R}$, we will define its H^k-Sobolev norm as follows:

$$\|f\|^2_{H^K(\mathbb{S}^3)} \doteq \sum_{a=0}^{k} \int_{\mathbb{S}^3} \left|\nabla^a f\right|^2_{g_{\mathbb{S}^3}} \mathrm{dvol}_{\mathbb{S}^3},$$

where $\nabla^a f$ denotes the $(0,a)$-type tensor with components $\nabla_{i^1} \dots \nabla_{i^a} f$. Show that there exists some constant $C > 0$ such that, for any function $f : \mathbb{R}^{3+1} \to \mathbb{R}$ with

$$\sum_{|\alpha|=0}^{k} \int_{t=0} \left|(1+|x|^2)^{|\alpha|} \partial_x^\alpha f\right|^2 dx = A,$$

we have

$$\|\Omega^{-1} f \circ \mathcal{F}^{-1}|_{T=0}\|^2_{H^k(\mathbb{S}^3)} \leqslant CA.$$

(e) Let us consider an initial value problem on $\mathbb{R} \times \mathbb{S}^3$ of the form

$$\begin{cases} \Box_{g_E} h(T,X) = F(T,X,h,dh), \\ (h|_{T=0}, \partial_T h|_{T=0}) = (h_0, h_1) \in H^k(\mathbb{S}^3) \times H^{k-1}(\mathbb{S}^3), \end{cases}$$

where $F : \mathbb{R} \times \mathbb{S}^3 \times \mathbb{R} \times T^*(\mathbb{R} \times \mathbb{S}^3) \to \mathbb{R}$ is a *smooth* function. Show that, for any $k > 2n+2$, there exists some $\epsilon > 0$ such that if $\|h_0\|^2_{H^k(\mathbb{S}^3)} + \|h_1\|^2_{H^{k-1}(\mathbb{S}^3)} < \epsilon$, then there exists a solution h on the whole of the domain $\{-\pi \leqslant T \leqslant +\pi\} \subset \mathbb{R} \times \mathbb{S}^3$. (*Hint: Aply the local well-posedness theory for general non-linear wave equations.*)

*(f) Show that the equation

$$\Box_\eta \phi = -(\partial_t \phi)^2$$

on $\mathbb{R}^{3+1}$, when reexpressed as an equation on $\mathcal{F}(\mathbb{R}^{3+1})$ in terms of $\tilde{\phi}$ has coefficients which become singular at $\mathcal{I}^+$ (see also the comment below exercise 12.1). On the other hand, check that the equation

$$\Box_\eta \phi = \eta(d\phi, d\phi) \tag{2}$$

transforms in an equation which is regular (i.e. is as in part (e)) on the whole of $\overline{\mathcal{F}(\mathbb{R}^{3+1})}$. Deduce that, for any fixed $k > 2n+2$, there exists an $\epsilon > 0$ such that for any smooth and compactly supported initial data set (ϕ_0, ϕ_1) for (2) at $t = 0$ satisfying

$$\sum_{|\alpha|=0}^{k} \int_{t=0} |(1+|x|^2)^{|\alpha|} \partial_x^\alpha \phi_0|_{t=0}|^2 \, dx + \sum_{|\alpha|=0}^{k-1} \int_{t=0} |(1+|x|^2)^{|\alpha|} \partial_x^\alpha \phi_1|_{t=0}|^2 \, dx < \epsilon,$$

the corresponding solution for (2) exists globally (i.e. on the whole of $\mathbb{R}^{3+1}$. (*Hint: Transform this to an initial value problem on* $\mathbb{R} \times \mathbb{S}^3$ *and use part (e).*)